MATH 137 PRACTICE FINAL EXAM

Solutions

Notes:

1. Answer all questions in the space provided. You may use the last page as additional space for solutions. Clearly mark this if you do.
2. Your grade will be influenced by how clearly you express your ideas and how well you organize your solutions. Show all details to get full marks. Numerical answers should be in exact values (no approximations). For example, $\frac{\sqrt{3}}{2}$ is acceptable, 0.8660 is not.
3. There are a total of ?? possible points.
4. Check that your exam has ?? pages, including the cover page.
5. DO NOT write on the Crowdmark QR code at the top of the pages or your exam will not be scanned (and will receive a grade of zero).
6. Use a dark pen or pencil.

(MC) Answer the following multiple choice questions by writing either a, b, c, or d in the box to the right of the question. Note there is only one correct answer for each question.

1. $\lim_{x \to 3} \ln|x - 3| =$ $\boxed{C}$
   (a) 0.
   (b) ∞.
   (c) $-\infty$.
   (d) None of the above.

2. If f is continuous on $[a, b]$ and differentiable on (a, b), then $\boxed{A.}$
   (a) for any $x_1, x_2 \in (a, b)$ where $x_1 < x_2$, there exists $c \in (x_1, x_2)$ so that $f'(c) = \dfrac{f(x_1) - f(x_2)}{x_1 - x_2}$.
   (b) $f(a) \leq f(x) \leq f(b)$ for all $x \in [a, b]$.
   (c) $f'(x)$ is continuous on (a, b).
   (d) None of the above.

3. $\lim_{x \to 0} \dfrac{\cos(x) + \sin(x) - 1}{x} = 1$ $\boxed{D}$
   (a) -1.
   (b) 0.
   (c) 2.
   (d) None of the above.

4. If $\lim_{x \to a} f(x) = L \in \mathbb{R}$ and $\{x_n\}$ is a sequence such that $x_n \to a$ as $n \to \infty$, and $x_n \neq a$ for all $n \in \mathbb{N}$, then $\boxed{C}$
   (a) $\lim_{n \to \infty} f(x_n)$ does not exist.
   (b) f is continuous at $x = a$.
   (c) $\lim_{n \to \infty} f(x_n) = L$.
   (d) None of the above.

5. For a function f and $a \in \mathbb{R}$, if $f''(a) = 0$ and $f'(a) = 0$, then $\boxed{B}$
   (a) $x = a$ is a point of inflection for f.
   (b) $x = a$ is a critical point of f.
   (c) f cannot have a local maximum at $x = a$.
   (d) None of the above.

(TF) True/False, answer in the box below the question by writing TRUE or FALSE.

6. TRUE or FALSE: For $a \in \mathbb{R}$, $|x - a| \leq 1$ defines a closed interval of length 1.
   $\boxed{\text{False}}$ length 2!

7. TRUE or FALSE: $f(x) = 3x^4 + 2x - 1$ has a root on $[0, 1]$.
   $\boxed{\text{TRUE}}$

8. TRUE or FALSE: If $f'(x) = \cos(x)$ then $f(x) = \sin(x)$. $f(x) = \sin(x) + C$.
   $\boxed{\text{False}}$

9. TRUE or FALSE: Let $a_n = f(n)$ where f is a continuous function defined on $\mathbb{R}$. If $\lim_{n \to \infty} a_n = L$ then $\lim_{x \to \infty} f(x) = L$. False, $f(x) = \sin(x\pi)$.
   $\boxed{\text{False}}$

10. TRUE or FALSE: If f is not differentiable at $x = a \in \mathbb{R}$, then for $k \in \mathbb{R}$, $g(x) = f(x) + k$ is not differentiable at $x = a$.
    $\boxed{\text{TRUE}}$

(SA) Short answer questions. marks only awarded for a correct final answer, you do not need to show any work. **Clearly indicate your final answer.**

1. For $f(x) = \ln(e + x)$, find $L_0^f(x)$.

$f(0) = 1$

$f'(x) = \frac{1}{e+x} \to f'(0) = \frac{1}{e} \quad L_0^f(x) = 1 + \frac{1}{e}(x) = 1 + \frac{x}{e}$

2. If f is a differentiable function such that $f(0) = 1$ and $f'(x) \in [1, 5]$ for all $x \in \mathbb{R}$, use the Bounded Derivative Theorem to write down an interval that $f(3)$ must lie in.

$f(0) + 1(3-0) \leq f(3) \leq f(0) + 5(3-0)$

$1 + 3 \leq f(3) \leq 1 + 5(3)$

$f(3) \in \boxed{[4, 16]}$

3. Give an example of a differentiable function f that is concave up everywhere, but $f''(0)$ does not exist.

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ x^4 & \text{if } x < 0 \end{cases}$$

4. Give an example of a function f that is differentiable on $(0, 1)$. both $f(0)$ and $f(1)$ are defined. but the Mean Value Theorem cannot be applied to f.

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1. \end{cases}$$

5. If $f(3) = 1$ and $f'(3) = \pi$, find $(f^{-1})'(1)$.

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(3)} = \frac{1}{\pi}$$

Long Answer

#1a) $\lim_{n\to\infty} \frac{\sin(n\pi)}{\sin(n)} = \lim_{n\to\infty} \frac{0}{\sin(n)} = 0$ ($\sin(n\pi)=0$ for all $n\in\mathbb{N}$).

b) $\lim_{n\to\infty} \frac{\sin(n)}{n^2+1}$. Note that $-1 \le \sin(n) \le 1 \Rightarrow \frac{-1}{n^2+1} \le \frac{\sin(n)}{n^2+1} \le \frac{1}{n^2+1}$

and $\lim_{n\to\infty} \frac{\pm 1}{n^2+1} = 0$.

So, by the Squeeze Theorem, $\lim_{n\to\infty} \frac{\sin(n)}{n^2+1} = 0$.

c) $\lim_{n\to\infty} \frac{n^3+n+1}{3n^3+n^2} = 1/3$

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2) Suppose $\{a_n\}$ is bounded, so $|a_n| \le M$ for some $M>0$.

Next, consider $|a_nb_n|$: $|a_nb_n| \le M|b_n|$, which means

$$-M|b_n| \le a_nb_n \le M|b_n|.$$

Since $\lim_{n\to\infty} b_n = 0$, we get $\lim_{n\to\infty} |b_n| = 0$, which means

$\lim_{n\to\infty} \pm M|b_n| = 0$ too. So, by the Squeeze Theorem,

$\lim_{n\to\infty} a_nb_n = 0$, as desired.

QED

#3a) $f(x) = x^2e^x\ln(x)$

$f'(x) = 2xe^x\ln(x) + x^2e^x\ln(x) + x^2e^x\cdot\frac{1}{x}$

b) $f(x) = \tan(\cos(x))$

$f'(x) = \sec^2(\cos(x))\cdot(-\sin x)$

#4) a) $\ln(x)+\ln(y) = xy$, use implicit diff:

$$\frac{1}{x} + \frac{y'}{y} = y + xy'$$

$$\Rightarrow \frac{y'}{y} - xy' = y - \frac{1}{x}$$

any of these are fine!

$$\Rightarrow y'\left(\frac{1}{y} - x\right) = y - \frac{1}{x}$$

$$\Rightarrow y' = \frac{y - 1/x}{\frac{1}{y} - x} = \frac{\left(\frac{xy-1}{x}\right)}{\left(\frac{1-xy}{y}\right)} = \frac{xy^2 - y}{x - x^2 y}$$

b) $y = (\sin(x))^{\ln(x)}$ use logarithmic diff:

$$\Rightarrow \ln(y) = \ln(x)\ln(\sin(x))$$

$$\therefore \frac{y'}{y} = \frac{\ln(\sin x)}{x} + \ln x \cdot \frac{\cos x}{\sin x}$$

$$\Rightarrow \boxed{y' = (\sin x)^{\ln(x)}\left[\frac{\ln(\sin x)}{x} + \frac{\ln x \cos x}{\sin x}\right]}$$

#5) $f(x) = (x-1)|x+2| - 3 = \begin{cases} (x-1)(x+2)-3 & \text{if } x \geq -2 \\ (x-1)(-x-2)-3 & \text{if } x < -2 \end{cases}$

$= \begin{cases} x^2+x-5 & \text{if } x \geq -2 \\ -x^2-x-1 & \text{if } x < -2 \end{cases}$

So, for $x \neq -2$, $f'(x) = \begin{cases} 2x+1 & \text{if } x \geq -2 \\ -2x-1 & \text{if } x < -2 \end{cases}$

At $x = -2$, $f'(-2) = \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h \to 0} \frac{(-2+h-1)|-2+h+2| - 3 + 3}{h}$

$= \lim_{h \to 0} (-3+h) \frac{|h|}{h}$ does not exist.

So $x = -2$ is a C.P. for f.

$f'(x) = 0$ if $x = -1/2$, so $x = -1/2$ is also a C.P.

Now, we test: $f(0) = -5$

$f(-1/2) = (-3/2)(3/2) - 3 = -\frac{9}{4} - 3 = -\frac{21}{4} = -5.25$

$f(-2) = -3$

$f(-3) = -7$

So the global max. is -3 (when $x = -2$) and the global min. is -7 (when $x = -3$).

#6) a) If f is continuous on $[a,b]$ and either $f(a) < \alpha < f(b)$ or $f(a) > \alpha > f(b)$, then there exists $c \in (a,b)$ such that $f(c) = \alpha$.

b) f is continuous, and $f(0) = 1 > 0$

$f(-1) = -3 < 0$

So there exists $c \in (-1,0)$ such that $f(c) = 0$ by the IVT. So the interval we want is $[-1, 0]$ (or $(-1,0)$).

c) $x_1 = 0$, $f(x) = x^3 + 3x + 1$, $f'(x) = 3x^2 + 3$

$$\therefore x_{n+1} = x_n - \frac{x_n^3 + 3x_n + 1}{3(x_n)^2 + 3}$$

So
$$x_2 = 0 - \frac{0^3 + 3(0) + 1}{3(0)^2 + 3} = -\frac{1}{3}$$

$$x_3 = -\frac{1}{3} - \frac{\left(-\frac{1}{3}\right)^3 + 3\left(-\frac{1}{3}\right) + 1}{3\left(-\frac{1}{3}\right)^2 + 3}$$

$$= -\frac{1}{3} - \frac{-\frac{1}{27} + -1 + 1}{\frac{1}{3} + 3}$$

$$= -\frac{1}{3} - \frac{\left(\frac{-1}{27}\right)}{\left(\frac{10}{3}\right)}$$

$$= -\frac{1}{3} - \frac{-1}{27} \cdot \frac{3}{10}$$

$$= -\frac{1}{3} - \frac{-1}{90} = -\frac{29}{90}$$

#7) a) $f'(x) = \frac{2x}{x^2+1} = 0$ if $x = 0$.

		0	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

f is increasing on $[0, \infty]$ and decreasing on $(-\infty, 0]$.

b) $f''(x) = \frac{(x^2+1)(2) - 2x(2x)}{(x^2+1)^2} = \frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2} = 0$

if ~~x~~ $x = \pm 1$.

		-1		1	
f''	$-$		$+$		$-$
f	$\cap$		$\cup$		$\cap$

So f is concave up on $[-1, 1]$ and concave down on $(-\infty, -1]$ and $[1, \infty)$.

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8) Suppose for a contradiction that $f(x_1) = f(x_2) = 0$ for $x_1 \neq x_2$. Say WLOG that $x_1 < x_2$. Apply the MVT to f on $[x_1, x_2]$, we get $c \in (x_1, x_2)$ so that $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$, but if $f'(c) = 0$ then c is a critical point of f, a contradiction to the fact that f has no critical points. So f must have at most one root.

QED.

#9 a) $\lim_{x\to 1^+} (\ln x)^{x-1}$ (type $(+0)^0$)

$= e^{\lim_{x\to 1^+} (x-1)\ln(\ln(x))}$ type $0 \cdot -\infty$

$= e^{\lim_{x\to 1^+} \frac{\ln(\ln(x))}{\frac{1}{x-1}}}$ type $\frac{-\infty}{\infty}$

$\stackrel{LHR}{=} e^{\lim_{x\to 1^+} \frac{\frac{1}{x\ln x}}{\frac{-1}{(x-1)^2}}} = e^{\lim_{x\to 1^+} \frac{-(x-1)^2}{x\ln x}}$ (type $0/0$)

$\stackrel{LHR}{=} e^{\lim_{x\to 1^+} \frac{-2(x-1)}{\ln x + 1}} = e^{0/1} = 1$

b) $\lim_{x\to 0^+} (\sqrt{x})^{\frac{1}{3\sqrt{x}}}$ (type 0^∞)

$= 0$ $(0^\infty \to 0)$.

c) $\lim_{x\to 0^+} (1+\sqrt{x})^{\frac{1}{3\sqrt{x}}}$ (type 1^∞)

$= e^{\lim_{x\to 0^+} \frac{\ln(1+\sqrt{x})}{3\sqrt{x}}}$ (type $0/0$)

$\stackrel{LHR}{=} e^{\lim_{x\to 0^+} \frac{\frac{1}{1+\sqrt{x}} \cdot \frac{1}{\cancel{2\sqrt{x}}}}{\frac{3}{\cancel{2\sqrt{x}}}}} = e^{\lim_{x\to 0^+} \frac{1}{3} \cdot \frac{1}{1+\sqrt{x}}} = e^{1/3}$

10) First, f needs to be continuous everywhere. Both $\sin(ax)$ and x^2+2x+b are continuous, so we check $x=0$:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} \sin(ax) = 0 = f(0)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2+2x+b = b.$$

So $b=0$

$$\therefore f(x) = \begin{cases} \sin(ax) & \text{if } x \geq 0 \\ x^2+2x & \text{if } x<0. \end{cases}$$

$$\text{So, } f'(x) = \begin{cases} a\cos(ax) & \text{if } x>0 \\ 2x+2 & \text{if } x<0 \end{cases}$$

So, we need to check what happens at $x=0$ again.

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{h^2+2h}{h} = \lim_{h\to 0^-} h+2 = 2.$$

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{\sin(ah)}{h} \cdot \frac{a}{a} = \lim_{h\to 0^+} \frac{\sin(ah)}{ah} \cdot a = a.$$

So we require $a=2$.

#11) a) Consider $h(x) = f(x) - g(x)$. Then, for $x \in I$,

$$h'(x) = f'(x) - g'(x) = 0$$

By the Constant Function Theorem, there exists $K \in \mathbb{R}$ so that $h(x) = K$ for $x \in I$.

But that means $f(x) - g(x) = K$ on I, or

$$f(x) = g(x) + K \text{ for } x \in I.$$

QED.

b) ~~Consider $h(x) = f(x) - g(x) - x^2$.~~

Since $f'(x) - g'(x) = 2x$, we get that $f'(x) = g'(x) + 2x$ on I.

$$= (g(x) + x^2)'$$

But by part (a), this means there is $K \in \mathbb{R}$ so that

$$f(x) = g(x) + x^2 + K \text{ on } I.$$

QED.

#12 a) $f(x) = \ln(1+x) \rightarrow f(0) = 0$

$f'(x) = \frac{1}{1+x} \rightarrow f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \rightarrow f''(0) = -1.$

So $T_{2,0}(x) = x - \frac{x^2}{2}$

b) $\ln(2) = \ln(1+1) \approx T_{2,0}(1) = 1 - \frac{1}{2} = \frac{1}{2}$

#12c) $f'''(x) = \frac{2}{(1+x)^3}$, so Taylor's Theorem says there is a c between 0 and x so that

$$f(x) - T_{2,0}(x) = \frac{f'''(c)}{3!}x^3 = \frac{2}{3!(1+c)^3}x^3 = \frac{x^3}{3(1+c)^3}.$$

d) For $x=1$, $\left|\frac{2}{(1+c)^3}\right| \leq 2$ for $c \in [0,1]$, as $f'''(c)$ is clearly decreasing.

so $|R_{2,0}(x)| \leq \frac{2}{3!}(x)^3 \Rightarrow |R_{2,0}(1)| \leq \frac{2(1)^3}{3!} = \frac{1}{3}$

(error)

e) for $x \geq 0$, $f(x) - T_{2,0}(x) = \frac{x^3}{3(1+c)^3} \geq 0$, so $f(x) \geq T_{2,0}(x)$, which means $f(1) \geq T_{2,0}(1)$, so the estimate is an underestimate.

f) $\ln(2) \in \left[T_{2,0}(x), T_{2,0}(x) + \text{error}\right] = \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{3}\right] = \boxed{\left[\frac{1}{2}, \frac{5}{6}\right]}$

13. Sketch the graph of $f(x)$, where

$$f(x) = \frac{(x-1)(x-4)}{(x-2)^2}, \qquad f'(x) = \frac{x+2}{(x-2)^3}, \qquad f''(x) = \frac{-2(x+4)}{(x-2)^4}.$$

Use this page for your work, on the next page you will summarize your findings and draw your graph. **Marks will be awarded to the next page only**. On your graph, label any intercepts, critical points, points of inflection, and asymptotes. In your summary, all points should include both x- and y-coordinates.

Domain: $x \neq 2$

Asymptotes: V.A. $x = 2$

$\lim_{x \to \pm\infty} f(x) = 1 \to$ H.A. $y = 1$. as $x \to \pm\infty$

y-int ($x=0$): $y = \frac{4}{4} = 1$

x-int ($y=0$): $x = 1, 4$

f' DNE @ $x = 2$, $f' = 0$ @ $x = -2$. $\left(-2, \frac{+18}{16}\right) = \left(-2, \frac{+9}{8}\right)$

f'' DNE @ $x = 2$, $f'' = 0$ @ $x = -4$. $\left(-4, \frac{+40}{36}\right) = \left(-4, \frac{10}{9}\right)$

		-4		-2		2	
f''	$+$		$-$		$-$		$-$
f'	$+$		$+$		$-$		$+$
f	∪ ↗		∩ ↗		∩ ↘		∩ ↗
Shape							
		I.P.		local max			

Summary:

Intercepts	Asymptotes	Critical Points	Inflection Points
$y=1$ $(0,1)$ $x=1,4$ $(1,0)$ $(4,0)$	$X=2$ $y=1$	$(-2, 9/8)$	$(-4, 10/9)$

The domain of f is: $X \neq 2$ or $(-\infty, 2) \cup (2, \infty)$

Intervals where f is increasing: $(-\infty, -2]$ and $(2, \infty)$

Intervals where f is decreasing: $[-2, 2)$

Intervals where f is concave up: $(-\infty, -4]$

Intervals where f is concave down: $[-4, 2)$, $(2, \infty)$